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THE SOLITARY WAVE AND PERIODIC WAVES IN SHALLOW WATER

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1. Introduction and discussion of results

In 1844 Scott Russell [1]* reported his experimental observations on the solitary wave, a wave consisting of a single elevation which propagates without change of form. (The wave motion is in two dimensions only.) Later Boussinesq [2] and Rayleigh [3] independently gave derivations of the approximate form and velocity-amplitude relationship for such a wave in shallow water. Then by a slight modification of Rayleigh's method, Korteweg and DeVries [4] obtained periodic waves of permanent type, which include the solitary wave as a special case when the wave length becomes infinite. Gwyther [5] obtained Rayleigh's results by a slightly different method, and McCowan [6] found more accurate results by guided guessing. The question of the existence of the solitary wave in water of finite depth has been considered by Weinstein [7], but the existence of such a wave has not yet been proved.

All the results mentioned above apply to irrotational, two-dimensional motion of an incompressible, inviscid fluid over a horizontal bottom. Rayleigh's treatment involves an iteration scheme of a peculiar kind and leads to a differential equation for the wave profile.

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* Numbers in square brackets refer to the bibliography at the end of the article.

The theory given by Boussinesq involves a number of physical assumptions in addition to those of the basic hydrodynamical theory; it also leads to a differential equation for the wave profile. Both of these methods assume that the depth of the water is small compared to some horizontal dimension, and they might be interpreted as developments of the whole problem in powers of the ratio of the depth to some horizontal dimension, such as wave length. However, because these procedures are so unsystematic, it is not clear that they are equivalent to such developments, nor to what order of approximation the solutions obtained are valid. The method of proceeding to higher approximations is also obscure.

The object of the present investigation is to discuss waves of permanent type* in shallow water by a method in which the character of the approximation is quite clear, and which is capable of being carried out to include terms of any desired order. The method consists in expanding the solution of the exact hydrodynamic problem systematically in powers of a dimensionless parameter $\sigma = \omega h$ where h is the depth of the undisturbed fluid and ω is the curvature at some point on the surface. The expansions are inserted into the equations of motion and the boundary conditions, and coefficients of like powers of σ are equated. The variables are chosen in such a way that the terms of zero order in the expansion in powers of σ satisfy the well-known equations of the nonlinear shallow water theory, which are analogous to the equations of gas dynamics.**

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* Actually stationary solutions are found; progressive waves may be obtained from them by adding a constant velocity to the fluid.

** These equations were first derived in this way by K. O. Friedrichs [9], to whom this method is due.

It is easily shown that the only solutions of these equations for the first approximation (satisfied by the terms independent of σ) which are of permanent form are the constant or piecewise constant (shock type) solutions. However, the equations for the second approximation (satisfied by the coefficients of σ) have solutions which yield periodic waves of permanent form, similar to those of Korteweg and DeVries, and also solitary waves, similar to those of Rayleigh and Boussinesq.

The solution of permanent form, as given by the first and second approximations, for the irrotational, two-dimensional motion of an incompressible, inviscid fluid of mean depth h over a horizontal bottom is

$$\eta = \eta_{min} + (\eta_{max} - \eta_{min})\,\mathrm{cn}^2[\tfrac{x}{\lambda}\, 2F_1(k)], \tag{1.1}$$

$$p = \rho g(\eta - y),$$

$$\bar{v} = 0,$$

$$\frac{\bar{u}}{\sqrt{gh}} = \frac{\eta_{max}}{h} - L - (\frac{\eta_{max}}{h} - \frac{\eta_{min}}{h})\,\mathrm{cn}^2[\tfrac{x}{\lambda}\, 2F_1(k)],$$

$$\frac{\lambda}{h} = \frac{4}{\sqrt{3}}\, F_1(k)(2L + 1 - \frac{\eta_{min}}{h})^{-1/2}.$$

In these equations η is the surface elevation, measured up from the bottom; x is the horizontal coordinate and y is the vertical distance above the bottom; η_{max} and η_{min} are the maximum and minimum surface elevations, respectively; λ is the wave length, p the pressure, ρ the fluid density, g the acceleration of gravity, $\bar{v}$ the vertical velocity, $\bar{u}$ the horizontal velocity; F_1 the complete elliptic integral of the first kind of modulus k; cn the Jacobi elliptic function of modulus k; L and k are certain parameters given by the last two equations of (5.24).

This solution yields a two-parameter family of periodic waves as does that of Korteweg and DeVries. The parameters η_{max} and η_{min} are subject only to the conditions

$$(1.2) \qquad 0 \leq \frac{\eta_{min}}{h} \leq 1, \quad \frac{\eta_{max}}{h} \geq 2 - \frac{\eta_{min}}{h},$$

which both follow from the definition of h or the relations among η_{max}, η_{min}, L, and k. For any values of η_{max} and η_{min} satisfying the inequalities above, the surface profile is periodic in x with the wave length λ. The height of the crest above the mean height is greater than the depth of the trough below the mean height. The crest is also narrower than the trough. The wave is thus not symmetric about the mean height as it is in the linear theory. This is shown in Figure 1, where one wave length of a typical wave profile is plotted.

Figure 2 shows the contour lines of $\frac{\lambda}{h}$ as a function of $\frac{\eta_{min}}{h}$ and $\frac{\eta_{max}}{h}$. From this figure it can be seen that λ increases as either η_{min} or η_{max} increases. The qualitative behavior of this function is the same as that of the function obtained by Korteweg and DeVries, but quantitatively it is slightly different. Thus, although formally the equation for the wave profile obtained by Korteweg and DeVries is the same as the first of equations (1.1), the two profiles are slightly different because of the different expressions for $\lambda(\eta_{max}, \eta_{min})$. Another difference between their solution and equations (1.1) occurs in the expression for $\bar{u}$, which they find to be independent of both x and y. From considerations of conservation of mass, that is much less reasonable than the dependence on x which is given by the fourth of equations (1.1). The second of equations (1.1) yields the pressure in the fluid, which is seen to be given by the

hydrostatic expression even in the second approximation. This expression therefore seems to be very accurate. In the Korteweg-DeVries solution the pressure is not obtained.

From Figure 2 it can be seen that, for a fixed value of $\frac{\eta_{min}}{h}$, as $\frac{\eta_{max}}{h}$ decreases, λ decreases until $\lambda = 0$ when $\frac{\eta_{max}}{h} = 2 - \frac{\eta_{min}}{h}$. In this limiting case the amplitude may be finite but the wave length is zero, so that the surface is everywhere discontinuous. On the other hand, if $\frac{\eta_{max}}{h}$ increases indefinitely, so does λ. While this occurs, the troughs become wider and the crests higher and relatively narrow.

It was hoped that the solution would impose an upper limit on $\frac{\eta_{max}}{h}$, and that for the limiting solution the surface slope would be discontinuous at the crests. Then, as shown by Stokes [8, p. 418], an angle of 120° would be formed at the crests. However, to the present order of approximation these results are not obtained. Further research leading to these results would certainly be worth while.

When $\frac{\eta_{max}}{h}$ and $\frac{\eta_{min}}{h}$ are nearly equal to one, both the Korteweg-DeVries solution and equations (1.1) reduce to the cosine solution of the linear shallow water theory. When $\frac{\eta_{min}}{h} = 1$ and $\frac{\eta_{max}}{h}$ is greater than one, but otherwise unrestricted, the wave length becomes infinite in both solutions. In this case equations (1.1) become (see Figure 3 for wave profile)

$$(1.3) \qquad \eta = h + (\eta_{max} - h)\ \mathrm{sech}^2\ \frac{x\sqrt{3}}{2h}\left(\frac{\eta_{max}}{h} - 1\right)^{1/2},$$

$$p = \rho g(\eta - y),$$

$$\bar{v} = 0,$$

$$\frac{\bar{u}}{\sqrt{gh}} = \frac{1}{2}(\eta_{max} + h) - (\eta_{max} - h)\ \mathrm{sech}^2\ \frac{x\sqrt{3}}{2h}\left(\frac{\eta_{max}}{h} - 1\right)^{1/2},$$

$$\lambda = \infty.$$

Equations (1.3) represent a solitary wave. The profile, given by the first of equations (1.3), is the same as that found by Boussinesq [2]. The Korteweg-DeVries solution reduces, in this case, to the solitary wave found by Rayleigh. This solution, for the profile, differs from the first of equations (1.3) only by having the additional factor $\sqrt{\frac{h}{\eta_{max}}}$ multiplying the argument of the hyperbolic secant. For waves of small amplitude this factor is nearly one, and thus in this case the Korteweg-DeVries-Rayleigh solution for the profile practically agrees with the first of equations (1.3). Both the Korteweg-DeVries-Rayleigh solution and the Boussinesq solution yield a horizontal velocity independent of x and y. As mentioned above, considerations of conservation of mass indicate that the dependence of the velocity on x as given by the fourth of equations (1.3) is more reasonable than the constant velocity. However, for x infinite, the fourth of equations (1.3) yields $\bar{u} = \sqrt{gh}\ \dfrac{\frac{\eta_{max}}{h} + 1}{2}$, and the velocity $\sqrt{g\eta_{max}}$ given by Korteweg-DeVries-Rayleigh agrees with this to first order in the relative amplitude $\dfrac{\eta_{max} - h}{h}$.

Thus, if the water at infinity is at rest, the propagation speeds of the wave, as given by the two solutions, agree to first order in the relative amplitude of the wave. However, according to the fourth of equations (1.3) the water under the crest would be moving, while in the Korteweg-DeVries-Rayleigh solution it would be stationary.

2. Derivation of the shallow water theory

In this section the exact equations of hydrodynamics and boundary conditions are employed. Viscosity and compressibility are not considered, and the bottom, given by $y = d(x,z)$, is assumed to be rigid. This implies that the normal component of velocity is zero on the bottom. The pressure $p(x,y,z)$ is assumed to be a constant (which is taken to be zero) on the surface $y = \eta(x,z)$. In addition, the motion is assumed to be irrotational. With these assumptions, and the kinematic condition that a particle on the surface remains on the surface, the equations of motion and boundary conditions are as follows (the y-axis is vertically upward; $\bar{u}$, $\bar{v}$, $\bar{w}$ are the x, y, z velocity components; ρ is the density and g the acceleration of gravity):

$$(2.1) \qquad \bar{u}_x + \bar{v}_y + \bar{w}_z = 0,$$

$$\bar{u}_t + \bar{u}\bar{u}_x + \bar{v}\bar{u}_y + \bar{w}\bar{u}_z = -p_x/\rho,$$

$$\bar{v}_t + \bar{u}\bar{v}_x + \bar{v}\bar{v}_y + \bar{w}\bar{v}_z = -p_y/\rho - g,$$

$$\bar{w}_t + \bar{u}\bar{w}_x + \bar{v}\bar{w}_y + \bar{w}\bar{w}_z = -p_z/\rho,$$

$$\bar{w}_y = \bar{v}_z, \qquad \bar{u}_z = \bar{w}_x, \qquad \bar{v}_x = \bar{u}_y,$$

$$\eta_t + \bar{u}\eta_x + \bar{w}\eta_z = \bar{v} \qquad \text{at } y = \eta,$$

$$p = 0 \qquad \text{at } y = \eta,$$

$$\bar{u}d_x - \bar{v} + \bar{w}d_z = 0 \qquad \text{at } y = d.$$

A transformation to dimensionless variables will now be performed. The constants h and ω, with dimensions of length and reciprocal length respectively, will be introduced. Later the dimensionless constant $\sigma = \omega^2 h^2$ will be an expansion parameter, and it is small when h, the typical length in the vertical direction, is small compared to ω^{-1}, the typical length in the horizontal direction. For this reason the equations obtained when σ is small are called the equations of the shallow water theory.

The transformation equations, defining the new dimensionless variables, are

$$(2.2)\qquad \begin{aligned} &x = \alpha\omega^{-1}, && \tau = \sqrt{gh}\,\omega t, && \bar{u} = \sqrt{gh}\,u, \\ &y = \beta h, && \eta = hY, && \bar{v} = \sqrt{gh}(\omega h)^{-1}v, \\ &z = \gamma\omega^{-1}, && d = hH, && \bar{w} = \sqrt{gh}\,w, \\ & && p = \rho g h\pi, && \sigma = \omega^2 h^2. \end{aligned}$$

It is essential that the scale factors in the horizontal and vertical directions are different. When these expressions are used in equations (2.1), one obtains

$$(2.3)\qquad \begin{aligned} \sigma u_\alpha + v_\beta + \sigma w_\gamma &= 0, \\ \sigma[u_\tau + uu_\alpha + wu_\gamma + \pi_\alpha] + vu_\beta &= 0, \\ \sigma[v_\tau + uv_\alpha + wv_\gamma + \pi_\beta + 1] + vv_\beta &= 0, \\ \sigma[w_\tau + uw_\alpha + ww_\gamma + \pi_\gamma] + vw_\beta &= 0, \\ w_\beta = v_\gamma, \quad u_\gamma = w_\alpha, \quad v_\alpha &= u_\beta, \\ \sigma[Y_\tau + uY_\alpha + wY_\gamma] &= v \quad \text{at } \beta = Y, \\ \pi &= 0 \quad \text{at } \beta = Y, \\ \sigma[uH_\alpha + wH_\gamma] &= v \quad \text{at } \beta = H. \end{aligned}$$

In order to solve these equations, it will be assumed that u, v, w, π, and Y can be written in power series in σ. These series will then be inserted into the above equations, and the coefficients of like powers of σ equated. The coefficient of σ^n in each series has the superscript n (e.g. u^n) and is a function of α, β, γ, and τ, except the coefficients Y^n which are independent of β.

Thus inserting the series and retaining terms of zero order in σ in the above equations, one has

$$(2.4) \qquad \begin{gathered} v^o_\beta = 0, \\ v^o u^o_\beta = 0, \\ v^o v^o_\beta = 0, \\ v^o w^o_\beta = 0, \\ w^o_\beta = v^o_\gamma, \quad u^o_\gamma = w^o_\alpha, \quad v^o_\alpha = u^o_\beta, \\ v^o = 0 \quad \text{at } \beta = Y^o, \\ \pi^o = 0 \quad \text{at } \beta = Y^o, \\ v^o = 0 \quad \text{at } \beta = H. \end{gathered}$$

These equations yield

$$(2.5) \qquad \begin{aligned} v^o(\alpha,\beta,\gamma,\tau) &= 0, \\ w^o &= w^o(\alpha,\gamma,\tau), \\ u^o &= u^o(\alpha,\gamma,\tau), \\ \pi^o(\alpha,Y^o,\gamma,\tau) &= 0. \end{aligned}$$

Now, returning to equations (2.3) and equating coefficients of the first power of σ to zero one obtains, after using equations (2.5),

$$u^o_\alpha + w^o_\gamma = -v^1_\beta, \tag{2.6}$$

$$u^o_\tau + u^o u^o_\alpha + w^o u^o_\gamma + \pi^o_\alpha = 0,$$

$$\pi^o_\beta + 1 = 0,$$

$$w^o_\tau + u^o w^o_\alpha + w^o w^o_\gamma + \pi^o_\gamma = 0,$$

$$Y^o_\tau + u^o Y^o_\alpha + w^o Y^o_\gamma = v^1 \quad \text{at } \beta = Y^o,$$

$$u^o H_\alpha + w^o H_\gamma = v^1 \quad \text{at } \beta = H.$$

The first of these equations can be integrated, yielding

$$v^1 = -(u^o_\alpha + w^o_\gamma)\beta + F(\alpha,\tau) \tag{2.7}$$

where $F(\alpha,\tau)$ can be determined from the last of equations (2.6). Thus, after determining $F(\alpha,\tau)$,

$$v^1 = -\beta(u^o_\alpha + w^o_\gamma) + (u^o H)_\alpha + (w^o H)_\gamma. \tag{2.8}$$

Similarly the third of equations (2.6) can be integrated and the integration constant determined from the last of equations (2.5). This gives the hydrostatic approximation

$$\pi^o(\alpha,\beta,\gamma,\tau) = Y^o(\alpha,\gamma,\tau) - \beta, \tag{2.9}$$

which is taken by Lamb [8, p. 254] as the starting point of the shallow water theory. Here this hydrostatic pressure relation is automatically satisfied by the solution

in the first approximation. After using equations (2.8) and (2.9) in the second, fourth, and fifth of equations (2.6), one has

$$u^o_\tau + u^o u^o_\alpha + w^o u^o_\gamma + Y^o_\alpha = 0, \tag{2.10}$$

$$w^o_\tau + u^o w^o_\alpha + w^o w^o_\gamma + Y^o_\gamma = 0,$$

$$Y^o_\tau + [u^o(Y^o - H)]_\alpha + [w^o(Y^o - H)]_\gamma = 0.$$

Equations (2.10) are the equations of the nonlinear shallow water theory for three-dimensional motion over an arbitrary bottom. The functions u^o, w^o, and Y^o depend upon α, γ, and τ, and the additional condition $u^o_\gamma = w^o_\alpha$ (from equations (2.4)) must be satisfied. The vertical velocity v^o is zero, and π^o is determined by equation (2.9).

3. The linear shallow water theory

The linear shallow water theory can be obtained from equations (2.10) by assuming that $Y^o = a + \varepsilon$, where a is constant and ε, u, w are small. If quadratic terms in these quantities are omitted, equations (2.10) yield

$$\varepsilon_{\tau\tau} = [\varepsilon_\alpha(a - H)]_\alpha + [\varepsilon_\gamma(a - H)]_\gamma. \tag{3.1}$$

This is the equation for the surface elevation in the linear shallow water theory. The velocities can be obtained from the equations $u^o_\tau = \varepsilon_\alpha$ and $w^o_\tau = \varepsilon_\gamma$ and the pressure from equation (2.9). The vertical velocity v^o is zero.

For two-dimensional motion (e.g. in a rectangular canal of constant width) w^o is zero and therefore u^o and Y^o are independent of γ. The nonlinear shallow water theory, equations (2.10), then becomes

$$(3.2) \qquad u^0_\tau + u^0 u^0_\alpha + Y^0_\alpha = 0,$$

$$Y^0_\tau + [u^0(Y^0 - H)]_\alpha = 0,$$

with v^0 still zero and π^0 given by equation (2.9). These are the well-known differential equations of the nonlinear shallow water theory, for two-dimensional motion, first obtained in this way by K. O. Friedrichs [9]. The linear shallow-water theory equation (3.1) becomes, for two-dimensional motion,

$$(3.3) \qquad \varepsilon_{\tau\tau} = [\varepsilon_\alpha (a - H)]_\alpha .$$

4. Higher approximations

In order to get more accurate solutions than those yielded by the ordinary shallow water theory, we continue the development in powers of σ. This is done in the present section for terms in σ^2 and σ^3, but only for the case of two-dimensional motion. First, considering terms in σ^2 in equations (2.3) and making use of the results obtained above for the first approximation, one has

$$(4.1) \qquad u^1_\alpha = -v^2_\beta,$$

$$u^1_\tau + u^0 u^1_\alpha + u^1 u^0_\alpha + \pi^1_\alpha = -v^1 u^1_\beta,$$

$$v^1_\tau + u^0 v^1_\alpha + \pi^1_\beta = -v^1 v^1_\beta,$$

$$u^1_\beta = v^1_\alpha,$$

$$Y^1_\tau + u^0 Y^1_\alpha + u^1 Y^0_\alpha - v^1_\beta Y^1 = v^2 \qquad \text{at } \beta = Y^0,$$

$$\pi^1 + \pi^0_\beta Y^1 = 0 \qquad \text{at } \beta = Y^0,$$

$$u^1 H_\alpha = v^2 \qquad \text{at } \beta = H.$$

The third and fourth equations above can be integrated if equation (2.8) is used for v^1. One finds

$$(4.2)\quad \pi^1 = \beta^2/2(u^o_{\alpha\tau} + u^o u^o_{\alpha\alpha} - u^{o^2}_{\alpha}) + \beta[u^o_\alpha(u^oH)_\alpha - u^o(u^oH)_{\alpha\alpha} - (u^o_\tau H)_\alpha] + g(\alpha,\tau),$$

$$(4.3)\quad u^1 = -\beta^2/2\, u^o_{\alpha\alpha} + \beta(u^oH)_{\alpha\alpha} + f(\alpha,\tau).$$

The functions $f(\alpha,\tau)$ and $g(\alpha,\tau)$ are so far undetermined. Using equation (4.3) in the first of equations (4.1) gives

$$(4.4)\quad v^2 = \frac{(\beta^3 - H^3)}{6} u^o_{\alpha\alpha\alpha} - \frac{(\beta^2 - H^2)}{2}(u^oH)_{\alpha\alpha\alpha} - (\beta - H)f_\alpha + u^1(\alpha,H,\tau)H_\alpha.$$

The integration constant in (4.3) has been determined from the last of equations (4.1).

From the sixth of equations (4.1), since $\pi^o_\beta = -1$, one has

$$(4.5)\qquad \pi^1 = Y^1 \quad \text{at } \beta = Y^o.$$

This equation yields $g(\alpha,\tau)$ in terms of Y^1 as

$$(4.6)\quad g(\alpha,\tau) = Y^1 - \frac{Y^{o^2}}{2}(u^o_{\alpha\tau} + u^o u^o_{\alpha\alpha} - u^{o^2}_\alpha) - Y^o[u^o_\alpha(u^oH)_\alpha - u^o(u^oH)_{\alpha\alpha} - (u^o_\tau H)_\alpha].$$

When equations (2.8), (4.2), and (4.3) are used in the second and fifth of equations (4.1), the following equations for $f(\alpha,\tau)$ and $Y^1(\alpha,\tau)$ are obtained:

$$(4.7) \qquad f_\tau + u^o f_\alpha + u^o_\alpha f = -g_\alpha - \tfrac{1}{2}[(u^o H)^2_\alpha]_\alpha ,$$

$$Y^1_\tau + u^o Y^1_\alpha + u^o_\alpha Y^1 = \frac{Y^{o^3} - H^3}{6} u^o_{\alpha\alpha\alpha} - \frac{Y^{o^2} - H^2}{2}(u^o H)_{\alpha\alpha\alpha} - (Y^o - H) f_\alpha$$

$$+ \left[\frac{Y^{o^2}}{2} u^o_{\alpha\alpha} - Y^o (u^o H)_{\alpha\alpha} - f\right](Y^o_\alpha - H_\alpha).$$

The term g_α is to be eliminated by using equation (4.6). When f and Y^1 are found from equations (4.7), equation (4.6) yields $g(\alpha,\tau)$ and equations (4.2) and (4.3) then give u^1 and π^1. Equation (2.8) already gives v^1. Equations (4.7) are the equations of the first correction to the shallow water theory and will be called the equations of the second approximation, the shallow water theory being the first approximation.

Now returning to equations (2.3) and equating to zero the coefficients of σ^3, one finds for the third approximation

$$(4.8) \qquad u^2_\alpha = -v^3_\beta ,$$

$$u^2_\tau + u^o u^2_\alpha + u^1 u^1_\alpha + u^o_\alpha u^2 + \pi^2_\alpha = -v^1 u^2_\beta - v^2 u^1_\beta ,$$

$$v^2_\tau + u^o v^2_\alpha + u^1 v^1_\alpha + \pi^2_\beta = -v^1 v^2_\beta - v^2 v^1_\beta ,$$

$$u^2_\beta = v^2_\alpha ,$$

$$v^3 + v^2_\beta Y^1 + v^1_\beta Y^2 = u^o Y^2_\alpha + u^1 Y^1_\alpha + u^2 Y^o_\alpha + Y^2_\tau + u^1_\beta Y^1 Y^o_\alpha \qquad \text{at } \beta = Y^o,$$

$$-Y^2 + \pi^1_\beta Y^1 + \pi^2 = 0 \qquad \text{at } \beta = Y^o,$$

$$u^2 H_\alpha = v^3 \qquad \text{at } \beta = H.$$

Equations (4.8) for the third approximation could be treated in the same way as equations (4.1), but this will not be done here.

5. Steady state solutions and waves of permanent shape

The equations for the first three approximations in the case of two-dimensional motion have been obtained. The steady state solutions of these equations will now be investigated. Such solutions yield progressive waves of permanent shape when they are referred to a moving coordinate system.

Equations (3.2) of the first approximation (or non-linear shallow water theory) can be integrated at once in the steady state. When integrated they are

$$(5.1) \qquad \frac{1}{2} u^{o2} + Y^o = e,$$

$$u^o(Y^o - H) = m,$$

where m and e are constants.

Inserting the expression for u^o obtained from the second equation into the first gives

$$(5.2) \qquad \frac{m^2}{2(Y^o - H)^2} + Y^o = e.$$

This is a cubic equation for Y^o in terms of H, and shows that if H is a constant (i.e. the bottom is horizontal) then the only continuous steady state is Y^o = constant and thus u^o = constant. Since the equation has one root for which $Y^o - H$ is negative, and since this is physically impossible, there are only two possible continuous steady states for given values of m and e. There are also discontinuous steady state solutions consisting of these two constant solutions pieced together.*

* There may be many points of discontinuity, and at each such point the solution changes from one of the two constant states to the other. However, if it is

The steady states given by the first approximation or shallow water theory having been obtained for the case of a horizontal bottom, we proceed to investigate the second approximation to these solutions. Equations (4.7) become in this case (from equation (4.3), $f = u^1$)

$$u^o u^1_\alpha + Y^1_\alpha = 0, \tag{5.3}$$

$$(Y^o - H)u^1_\alpha + u^o Y^1_\alpha = 0.$$

The only solution of these homogeneous linear equations is $u^1_\alpha = Y^1_\alpha = 0$ and therefore u^1 = constant, Y^1 = constant, unless the determinant of the coefficients is zero. The determinant is zero if $(u^o)^2 = Y^o - H$, and in this case $Y^1_\alpha = -u^o u^1_\alpha$ where u^1_α is arbitrary. This condition means that $\bar{u}^o = \sqrt{g \cdot \text{depth}}$, which is the critical speed or propagation speed in the shallow water theory. Integrating and using equations (4.2) and (4.6) for π^1 and equation (2.8) for v^1 gives the solution

$$v^1 = 0, \quad \pi^1 = Y^1 = -u^o u^1 + c \quad \text{if } (u^o)^2 = Y^o - H \tag{5.4}$$

where u^1 is so far an arbitrary function of α and c is a constant.

The arbitrary function which appears in the second approximation can be determined by considering the third approximation. (This circumstance seems to be rather typical of perturbation solutions of boundary value problems.) Thus the steady state solutions of equations

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assumed that the energy of a small element of fluid does not increase on crossing the discontinuity, then only one discontinuous solution with a single point of discontinuity is possible. For this solution, the flow is from the shallower to the deeper side.

(4.8) with H, u^o, and Y^o constant and the second approximation given by equations (5.4) must be investigated. From equation (4.4) one has

$$(5.5) \qquad v^2 = -(\beta - H)u^1_\alpha .$$

Using this in the second and fourth of equations (4.8) gives

$$(5.6) \qquad \pi^2 = \frac{-u^o(\beta - H)}{2} u^1_{\alpha\alpha} + s(\alpha),$$

$$\iota^2 = \frac{-(\beta - H)^2}{2} u^1_{\alpha\alpha} + r(\alpha).$$

The sixth of equations (4.8) then yields

$$(5.7) \qquad Y^2 = \pi^2(\alpha, Y^o) = \frac{-u^o(Y^o - H)^2}{2} u^1_{\alpha\alpha} + s(x).$$

If equations (5.6) are used in the second of equations (4.8), the resulting equation may be integrated giving

$$(5.8) \qquad s + u^o r = -\frac{1}{2}(u^1)^2 + b$$

where b is constant. Similarly, one obtains from the third of equations (4.8)

$$(5.9) \qquad v^3 = \frac{u^1_{\alpha\alpha\alpha}}{6}(\beta - H)^3 - r_\alpha(\beta - H)$$

where the integration constant has been determined from the last of equations (4.8). When equation (5.9) is used in the fifth of equations (4.8), the resulting equation may be integrated, yielding

$$(5.10) \qquad u^o Y^2 + u^1 Y^1 = -(Y^o - H)r + (Y^o - H)^3 \frac{u^1_{\alpha\alpha\alpha}}{6} + j$$

where j is a constant.

Equations (5.7), (5.8), and (5.10) are inhomogeneous linear algebraic equations for the determination of Y^2, r, and s. However, the determinant of the coefficients vanishes (because $(u^o)^2 = Y^o - H$) and therefore the terms independent of the unknowns must satisfy a condition in order that the equations be consistent. The condition may be obtained by multiplying equations (5.7) and (5.8) by $-u^o$ and then adding equations (5.7), (5.8), and (5.10). The result, after Y^1 is eliminated by use of equation (5.4), is

$$(5.11) \quad \frac{-(Y^o - H)^3}{3} u^1_{\alpha\alpha} + \frac{3}{2} u^o (u^1)^2 - cu^1 + (j - u^o b) = 0.$$

This condition determines $u^1(\alpha)$, which was not determined by the equations of the second approximation. When $u^1(\alpha)$ is found the solution of the second approximation will be complete, and equations (5.7), (5.8), and (5.9) for the third approximation can be solved. However, an arbitrary function will appear in this solution, and it would be necessary to go to the fourth approximation to determine it.

Equation (5.11) is of the same type as that obtained by Rayleigh and Korteweg and DeVries, and may be integrated if it is multiplied by $\dfrac{6u^1_\alpha}{(Y^o - H)^3}$, and the result is

$$(5.12) \quad (u^1_\alpha)^2 = \frac{3u^o}{(Y^o-H)^3}(u^1)^3 - \frac{3c}{(Y^o-H)^3}(u^1)^2 + \frac{6(j-u^o b)}{(Y^o - H)^3} u^1 + q$$

where q is a constant. This equation may be integrated in terms of elliptic functions. To this end, let the roots of the cubic on the right be ℓ, h_1, and h_2 where $\ell \geq h_1 \geq h_2$ if all roots are real. If the roots are not all real, equation (5.12) has no non-constant solutions which are

bounded in the region $-\infty \le \alpha \le +\infty$. Since the sum of the roots is the negative of the coefficient of $(u^1)^2$ divided by that of $(u^1)^3$, the roots must satisfy the condition

$$\ell + h_1 + h_2 = \frac{c}{u^o} . \tag{5.13}$$

Now introduce the new variable χ by

$$u^1 = h_1 + (h_2 - h_1) \cos^2 \chi . \tag{5.14}$$

Then equation (5.12) becomes

$$\frac{d\chi}{d\alpha} = \Delta^{-1}\sqrt{1 - k^2 \sin^2 \chi} \tag{5.15}$$

where

$$\Delta^{-1} = \frac{1}{2}\sqrt{\frac{3}{(u^o)^5}(\ell - h_2)} , \qquad k^2 = \frac{h_1 - h_2}{\ell - h_2} . \tag{5.16}$$

Integrating equation (5.15),

$$\alpha - \alpha_o = \Delta \int_0^{\chi} (1 - k^2 \sin^2 \chi)^{-1/2} d\chi . \tag{5.17}$$

The origin may be chosen so that $\alpha_o = \chi_o = 0$, which implies that the origin occurs where $u^1 = h_2$. Thus $\alpha = \Delta F(\chi,k)$ where $F(\chi,k)$ is the elliptic integral of the first kind of modulus k. The inverse function is $\chi = \text{arc cos cn} \frac{\alpha}{\Delta} (\text{mod } k)$ and thus

$$u^1 = h_1 - (h_1 - h_2)\text{cn}^2 \frac{\alpha}{\Delta}(\text{mod } k). \tag{5.18}$$

This solution is periodic in α if ℓ, h_1, and h_2 are real, and the wave length is then

$$\lambda = 2\Delta \int_0^{\pi/2} (1 - k^2 \sin^2 \chi)^{-1/2}\, d\chi \tag{5.19}$$

$$= 2\Delta F(\tfrac{\pi}{2}, k) = 2\Delta F_1(k).$$

If the roots are not all real, the solution becomes infinite as α approaches infinity. For this reason only the case of real roots will be discussed.

From equations (5.4) and (5.18) one has

$$\pi^1 = Y^1 = c - u^0 h_1 + u^0(h_1 - h_2)\,\mathrm{cn}^2 \tfrac{\alpha}{\Delta}(\mathrm{mod}\ k). \tag{5.20}$$

The constant c may be determined if it is required that the mass of fluid in a wave length equal $\lambda(Y^0 - H)$, the amount of fluid per distance λ in the first approximation. Then $\int_0^{\lambda} Y^1 d\alpha = 0$, which leads to

$$c = u^0 \ell - u^0(\ell - h_2)\frac{E_1(k)}{F_1(k)} = u^0(\ell + h_1 + h_2). \tag{5.21}$$

Here $E_1(k)$ is the complete elliptic integral of the second kind, and the last equality is a consequence of equation (5.13).

The second approximation has now been determined and it has led to a two-parameter family of solutions. Now, combining the first and second approximations, one has

$$v = v^0 + \sigma v^1 = 0, \tag{5.22}$$

$$u = u^0 + \sigma u^1 = u^0 + \sigma h_1 - (\sigma h_1 - \sigma h_2)\,\mathrm{cn}^2 \tfrac{\alpha}{\Delta}(\mathrm{mod}\ k),$$

$$Y = Y^0 + \sigma Y^1 = Y^0 + u^0(\sigma \ell + \sigma h_2) + u^0(\sigma h_1 - \sigma h_2)\,\mathrm{cn}^2 \tfrac{\alpha}{\Delta}(\mathrm{mod}\ k),$$

$$\pi = \pi^0 + \sigma \pi^1 = Y^0 - \beta + u^0(\sigma \ell + \sigma h_2) + u^0(\sigma h_1 - \sigma h_2)\,\mathrm{cn}^2 \tfrac{\alpha}{\Delta}(\mathrm{mod}\ k).$$

Since $cn^2 \frac{\alpha}{\Delta}$ varies between zero and one, the maximum and minimum values of Y are given by

(5.23) $$Y_{max} = Y^o + u^o(\sigma\ell + \sigma h_1),$$

$$Y_{min} = Y^o + u^o(\sigma\ell + \sigma h_2).$$

These equations will be used to eliminate σh_1 and σh_2 from equations (5.22). Also it will be assumed that $H = 0$, which means that vertical distances are measured from the bottom, and that $Y^o = 1$, which implies that h is the average height of the surface. Then, upon reintroducing the original coordinates, equations (5.22), (5.21), and (5.19) become (the substitution $L = \sigma\ell$ has been used)

(5.24) $$\bar{v} = 0,$$

$$\frac{\bar{u}}{\sqrt{gh}} = \frac{\eta_{max}}{h} - L - \left(\frac{\eta_{max}}{h} - \frac{\eta_{min}}{h}\right) cn^2\left[\frac{x}{\lambda}\, 2F_1(k)\right] (\text{mod } k),$$

$$\eta = \eta_{min} + (\eta_{max} - \eta_{min}) cn^2\left[\frac{x}{\lambda}\, 2F_1(k)\right],$$

$$\frac{p}{\rho q} = \eta - y,$$

$$\frac{\lambda}{h} = \frac{4}{\sqrt{3}} F_1(k)\left(2L + 1 - \frac{\eta_{min}}{h}\right)^{-1/2},$$

where

$$2L + 1 > \frac{\eta_{max}}{h} > \frac{\eta_{min}}{h}, \qquad 0 < k^2 = \frac{\frac{\eta_{max}}{h} - \frac{\eta_{min}}{h}}{2L + 1 - \frac{\eta_{min}}{h}} \leq 1,$$

and

$$\left(2L + 1 - \frac{\eta_{min}}{h}\right) E_1(k) = \left(2L - 2 - \frac{\eta_{max}}{h} - \frac{\eta_{min}}{h}\right) F_1(k).$$

Equations (5.24) represent the steady state solutions as given by the first and second approximations. These

solutions have already been discussed in the introduction. When $\lambda = \infty$, equations (5.24) yield the solitary waves given by

$$(5.25) \qquad \bar{v} = 0, \qquad \lambda = \infty,$$

$$\frac{\bar{u}}{\sqrt{gh}} = \frac{1}{2}\left(\frac{\eta_{max}}{h} + 1\right) - \left(\frac{\eta_{max}}{h} - 1\right) \operatorname{sech}^2 \frac{x\sqrt{3}}{2h}\left(\frac{\eta_{max}}{h} - 1\right)^{1/2},$$

$$\frac{\eta}{h} = 1 + \left(\frac{\eta_{max}}{h} - 1\right) \operatorname{sech}^2 \frac{x\sqrt{3}}{2h}\left(\frac{\eta_{max}}{h} - 1\right)^{1/2},$$

$$\frac{p}{\rho g} = \eta - y.$$

This solution has also been discussed in the introduction.

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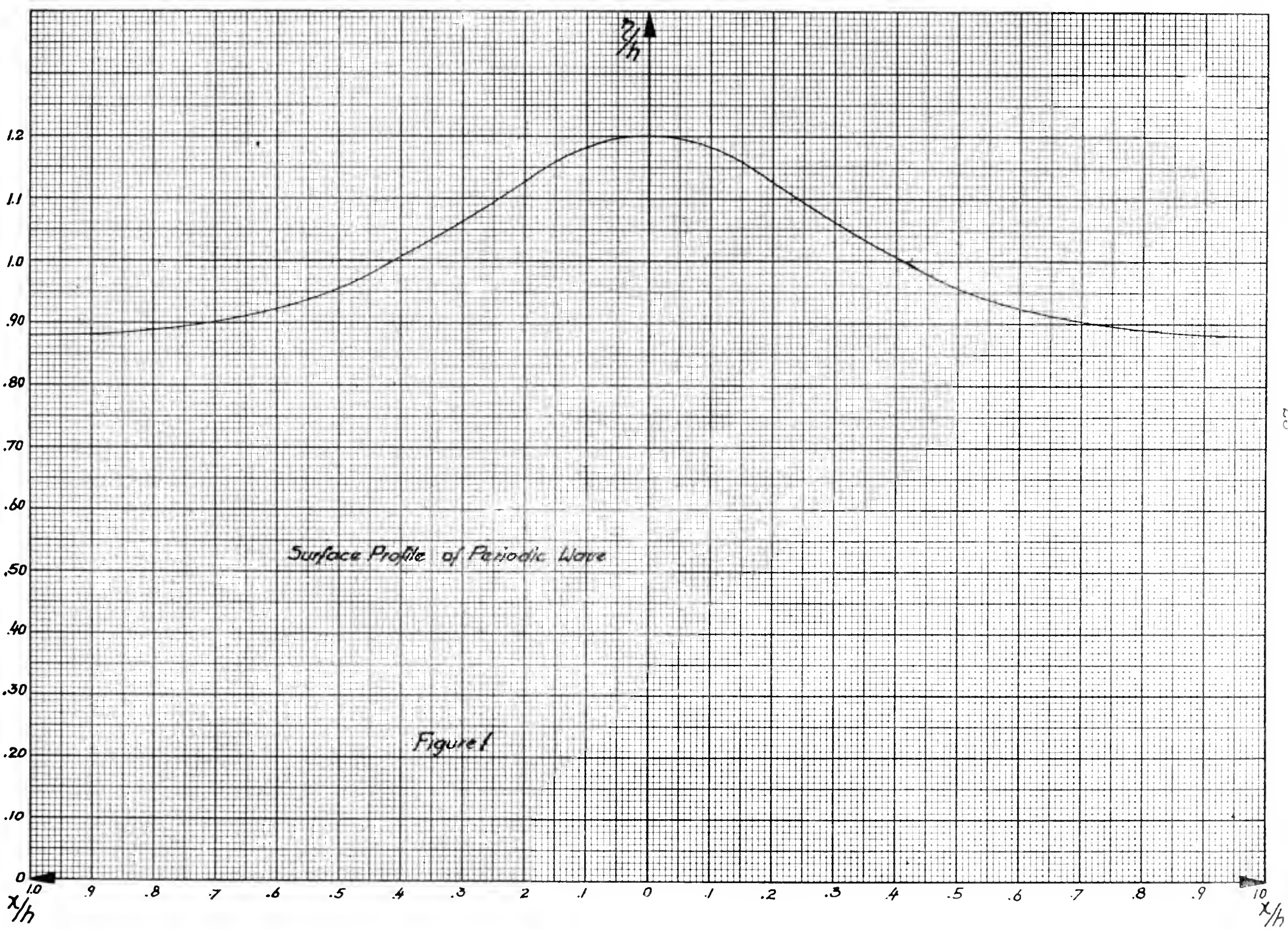


Surface Profile of Periodic Wave

Figure 1

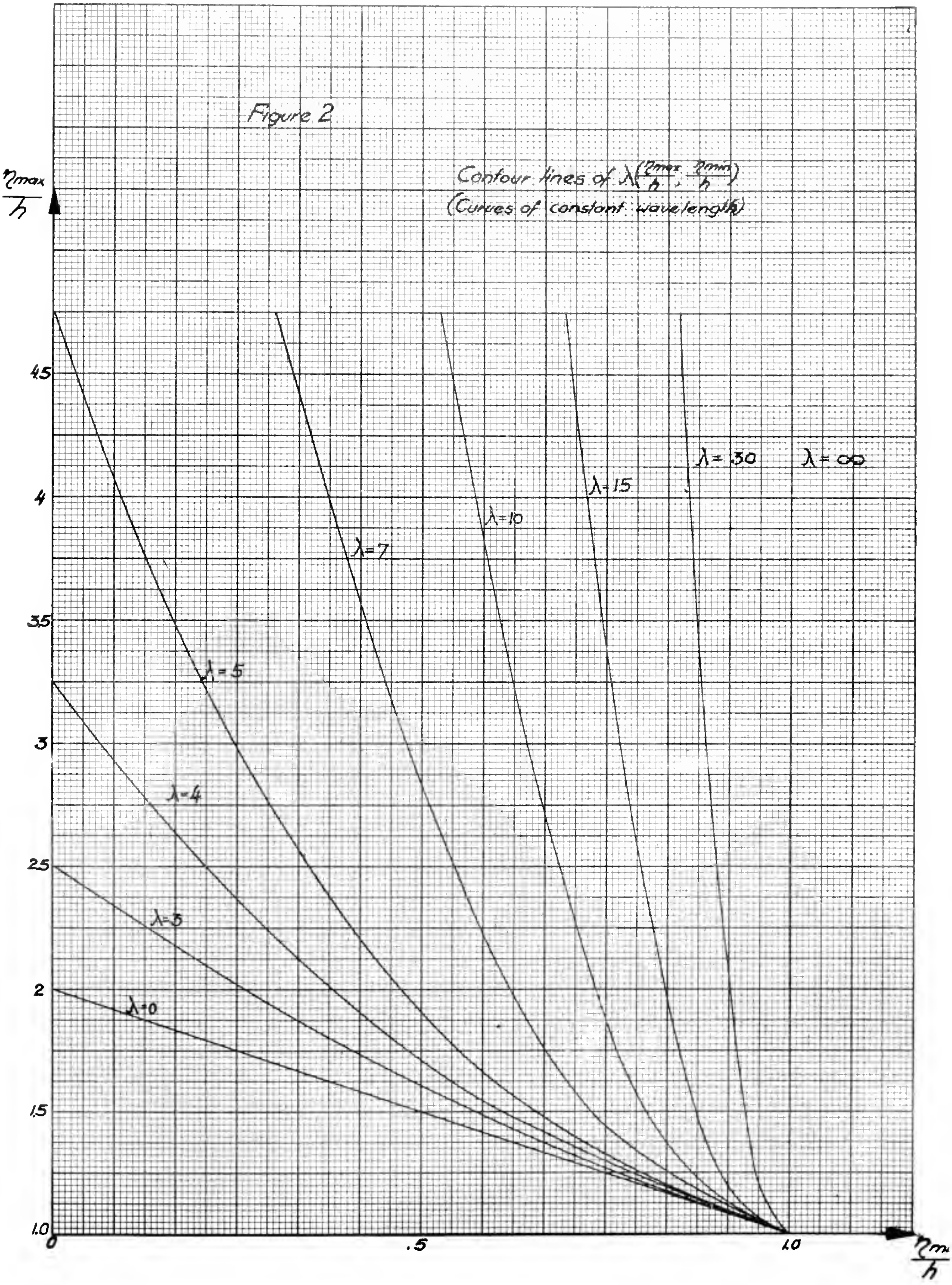
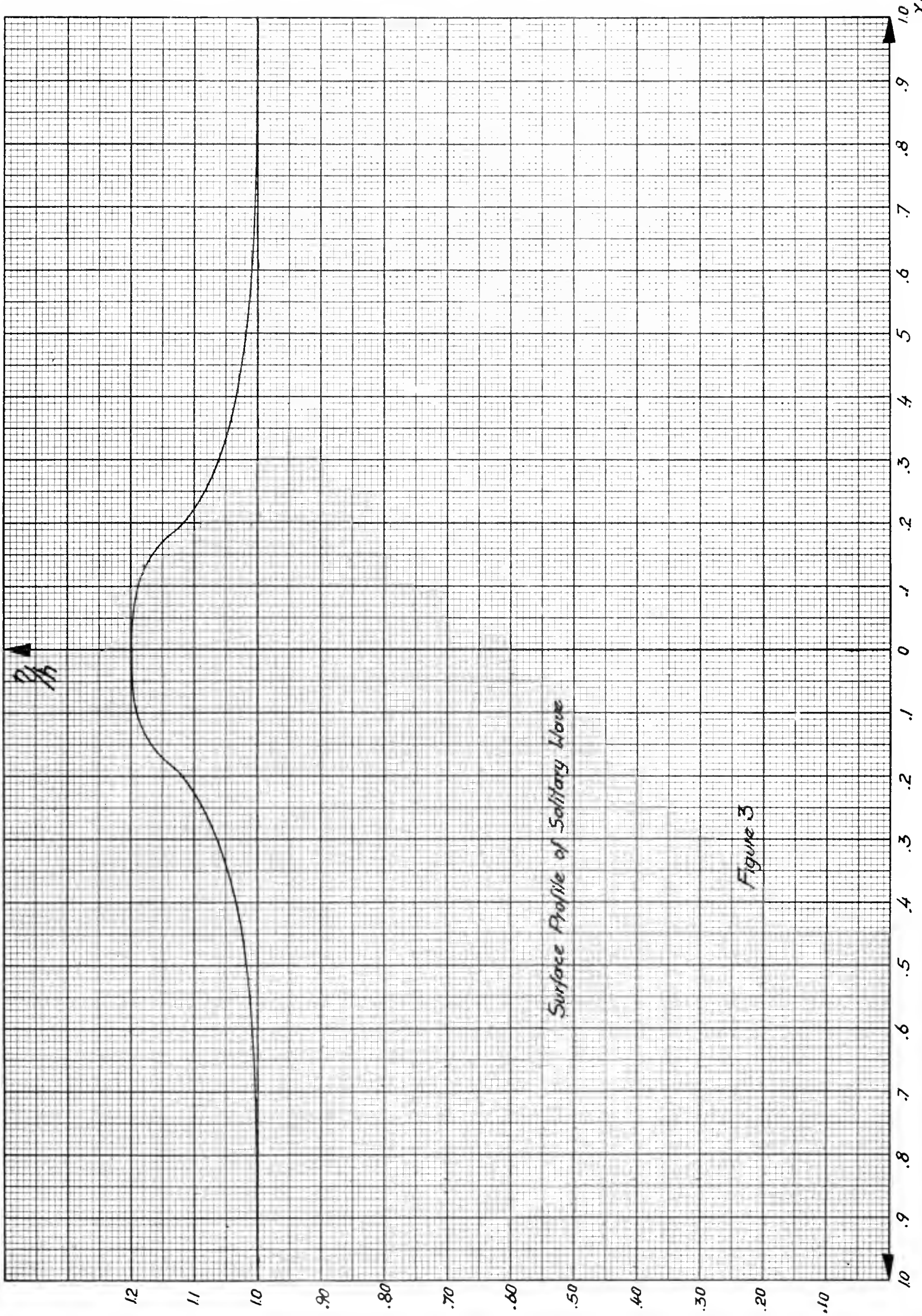


Figure 2
Contour lines of $\lambda\left(\frac{\eta_{max}}{h}, \frac{\eta_{min}}{h}\right)$
(Curves of constant wavelength)
$\frac{\eta_{max}}{h}$
4.5
4
3.5
3
2.5
2
1.5
1.0
0
.5
1.0
$\frac{\eta_{min}}{h}$
$\lambda = 30$
$\lambda = \infty$
$\lambda = 15$
$\lambda = 10$
$\lambda = 7$
$\lambda = 5$
$\lambda = 4$
$\lambda = 3$
$\lambda = 0$



Surface Profile of Solitary Wave

Figure 3

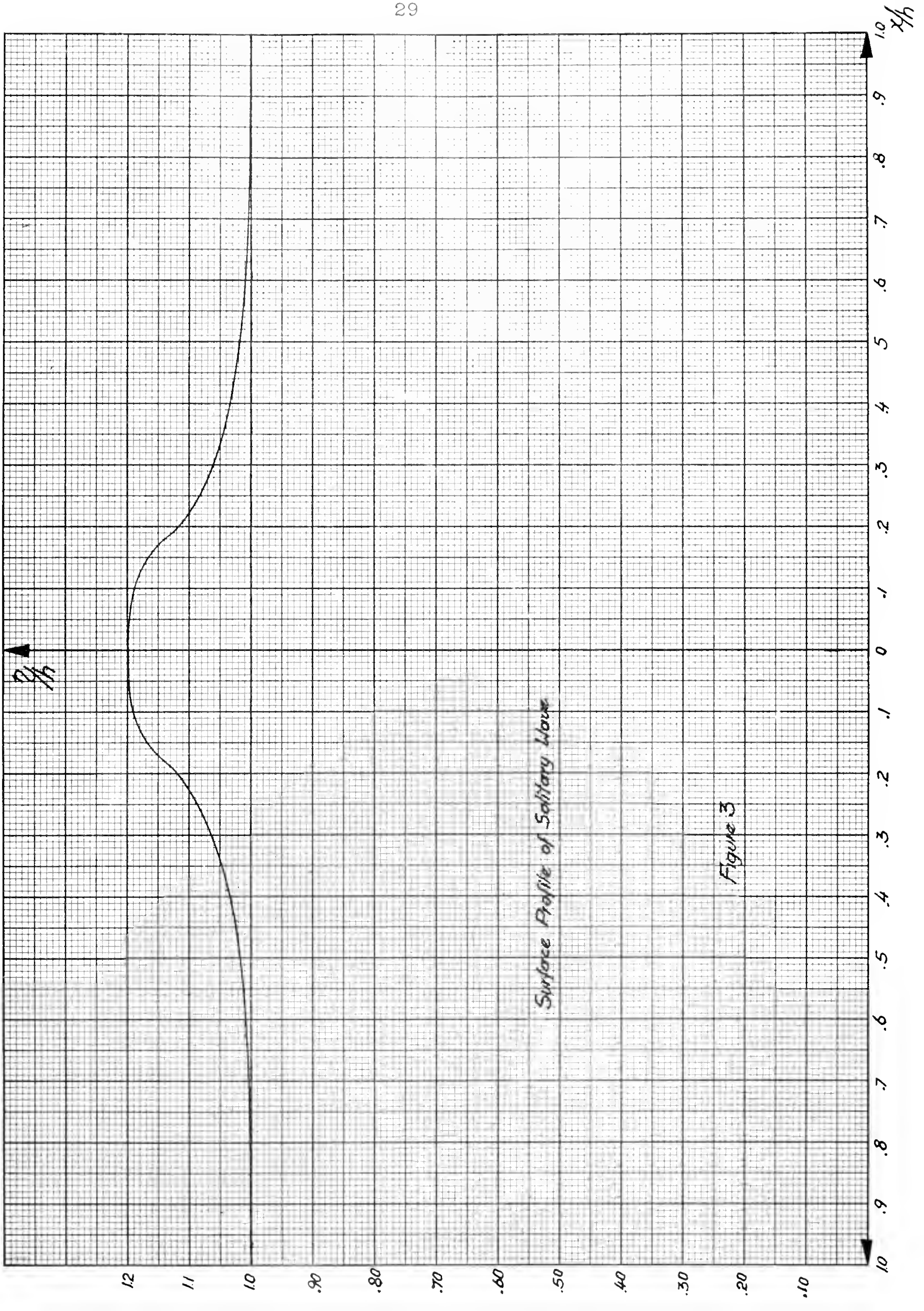

Surface Profile of Solitary Wave
x/h
η/h
1.0 .9 .8 .7 .6 .5 .4 .3 .2 .1 0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
1.2 1.1 1.0 .90 .80 .70 .60 .50 .40 .30 .20 .10

Figure 3

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